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Eigenvalues of elliptic operators and singular perturbations

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1 INTRODUCTION

This is a joint work with Professor Shuichi Jimbo of Hokkaido University.

We deal with the following eigenvalue problem:

$$\frac{1}{a_\zeta}\operatorname{div}(a_\zeta \nabla\Phi) + \mu\,\Phi = 0 \text{ in } \Omega, \quad \Phi = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Here a_ζ is a real valued step function of the form

$$a_\zeta(x) = \begin{cases} 1 & \text{for } x \in \overline{\Omega}_1, \\ \zeta & \text{for } x \in \Omega_2, \end{cases} \tag{1.2}$$

where $\zeta > 0$ is a perturbation parameter and Ω_2 is a subdomain $\Omega_2 \Subset \Omega$ with a smooth boundary and $\Omega_1 = \Omega \setminus \overline{\Omega}_2$. The boundary of Ω is denoted by Γ_1 and the one of Ω_2 is denoted by Γ_2 (see Figure 1). The coefficient a_ζ is discontinuous through Γ_2, we naturally consider (1.1) in a generalized sense, namely, $\mu \in \mathbb{C}$ is an eigenvalue of (1.1) if there exists $\Phi \in H_0^1(\Omega)$ such that $\Phi \not\equiv 0$ and

$$\int_\Omega (\nabla\Phi\nabla\varphi - \mu\,\Phi\,\varphi)\,a_\zeta\,dx = 0 \quad \text{for any } \varphi \in H_0^1(\Omega). \tag{1.3}$$

From a standard argument of self-adjoint operators, the eigenvalues of (1.3) are positive real numbers and the set of all eigenvalues is discrete and the system of all eigenfunctions spans $L^2(\Omega)$. The purpose of this paper is to characterize the limit of the eigenvalue $\mu_n(\zeta)$ of (1.3) as $\zeta \to 0$ and to find an approximation formula of $\mu_n(\zeta)$ at $\zeta = 0$. We will show that $\mu_n(\zeta)$ converges to an eigenvalue of the Laplacian on Ω_1 or Ω_2 (cf. Theorem 1.6) and the second coefficient of the asymptotic expansion of $\mu_n(\zeta)$ is an eigenvalue of a certain matrix (cf. Theorems 1.12, 1.13 and 1.16).

The problems of the form (1.1) are simplified eigenvalue problems of $-\Delta$ on a thin domain of $\mathbb{R}^{N+1}$ with a variable thickness. The coefficient a_ζ means

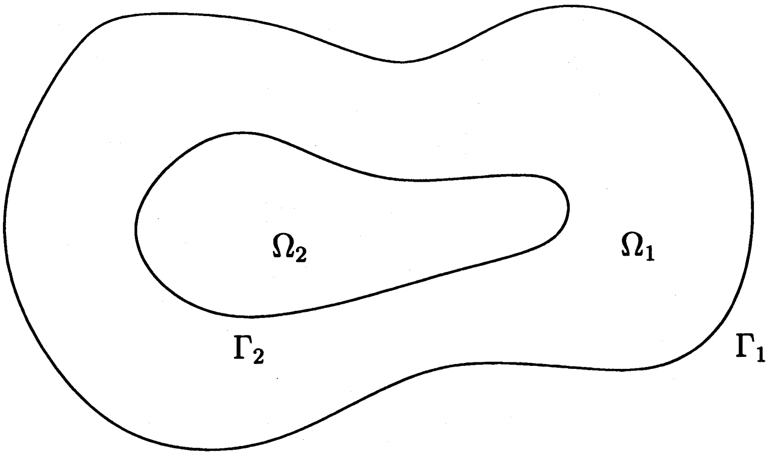

Figure 1: $\Omega = \Omega_1 \cup \overline{\Omega}_2$

a non-uniform thickness of the thin domain with the bottom Ω. We consider that an analysis of these operators give an understanding of characteristics of thin domains. The above type of elliptic differential operators also arise in some problems of the material science of non-uniform media. We mention some related works on perturbation of eigenvalues. Panasenko [11] studied the operator

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^{(\zeta)} \frac{\partial}{\partial x_j}\right)$$

where the discontinuous coefficients $a_{ij}^{(\zeta)}$ remain bounded in some subdomain and approach ∞ in the complement. He proved the set of eigenvalues converges to that of zeros of a certain equation with precise characterization. Our situation is similar to the above in a sence of that the coefficients of the operator are discontinuous and perturbed singularly. However, there are important differences in results. In our case, roughly speaking, eigenvalues are divided into two classes and we give an analysis on not only the asymptotics of each class but also the interaction between two classes.

Beale [1] and Jimbo [7, 8] characterize eigenvalues of the Laplacian subject to the Neumann boundary condition on domains which have a very thin channel and the channel degenerates into a line. They showed that the influence of the channel upon the eigenvalues dose not vanish in spite of the degeneration of the volume of the channel. Similar phenomena will occur in our situation.

Hereafter we prepare some notation to state our main results.

Notation 1.1 *Let* $\mu_n(\zeta)$ $(n = 1, 2, \dots)$ *be the eigenvalues of* (1.3) *arranged in increasing order with counting multiplicity.*

As it is mentioned above, the eigenvalues $\mu_n(\zeta)$ are real numbers. Without loss of generality the eigenfunctions can be taken to be real valued. Hereafter all functions appearing in this paper are also real valued. From a standard mini-max principle, the n-th eigenvalue $\mu_n(\zeta)$ is given by

$$\mu_n(\zeta) = \sup_{\substack{Y \subset L^2(\Omega) \\ \dim Y \leq n-1}} \inf_{\substack{\Phi \in H_0^1(\Omega) \\ \Phi \perp^{\zeta} Y}} R_\zeta(\Phi) \tag{1.4}$$

where $\Phi \perp^{\zeta} Y$ means

$$\int_\Omega \Phi \, \Psi \, a_\zeta \, dx = 0 \quad \text{for any } \Psi \in Y$$

and $R_\zeta(\Phi)$ denotes the Rayleigh quotient of Φ, i.e.,

$$R_\zeta(\Phi) = \int_\Omega |\nabla \Phi|^2 \, a_\zeta \, dx \Big/ \int_\Omega |\Phi|^2 \, a_\zeta \, dx.$$

Notation 1.2 *Let $\{\Phi_{n,\zeta}\}_{n=1}^{\infty}$ be a complete system of orthonormalized eigenfunctions of* (1.3)*, that is, $\Phi_{n,\zeta} \in H_0^1(\Omega)$ $(n = 1, 2, \dots)$ satisfy*

$$\int_\Omega (\nabla \Phi_{n,\zeta} \nabla \varphi - \mu_n(\zeta) \, \Phi_{n,\zeta} \, \varphi) \, a_\zeta \, dx = 0 \quad \textit{for any } \varphi \in H_0^1(\Omega), \tag{1.5}$$

$$\int_\Omega \Phi_{n,\zeta} \, \Phi_{m,\zeta} \, a_\zeta \, dx = \delta_{nm} \tag{1.6}$$

where δ_{nm} means Kronecker's delta symbol.

We relate the elaborate asymptotics of $\mu_n(\zeta)$ to the following eigenvalue problems given respectively on Ω_1 and Ω_2.

Notation 1.3 *Let ω_p $(p = 1, 2, \dots)$ be the eigenvalues arranged in increasing order (counting multiplicity) of the eigenvalue problem*

$$\Delta \phi + \omega \phi = 0 \textit{ in } \Omega_1, \quad \frac{\partial \phi}{\partial \nu_1} = 0 \textit{ on } \Gamma_2, \quad \phi = 0 \textit{ on } \Gamma_1 \tag{1.7}$$

where ν_1 is the unit outward normal vector on $\partial \Omega_1 = \Gamma_1 \cup \Gamma_2$ and let $\{\phi_p\}_{p=1}^{\infty}$ be a complete system of corresponding orthonormalized eigenfunctions, namely, the pair ϕ_p and ω_p satisfies (1.7) *and*

$$\int_{\Omega_1} \phi_p \, \phi_{p'} \, dx = \delta_{pp'} \quad (p, p' \geq 1).$$

Notation 1.4 *Let λ_q $(q = 1, 2, \dots)$ be the eigenvalues arranged in increasing order (counting multiplicity) of the following eigenvalue problem:*

$$\Delta\psi + \lambda\psi = 0 \text{ in } \Omega_2, \quad \psi = 0 \text{ on } \Gamma_2. \tag{1.8}$$

and $\{\psi_q\}_{q=1}^{\infty}$ a complete system of corresponding orthonormalized eigenfunctions, namely, the pair ψ_q and λ_q satisfies (1.8) *and*

$$\int_{\Omega_2} \psi_q \psi_{q'}\, dx = \delta_{q q'} \quad (q, q' \geq 1).$$

Notation 1.5 *We rearrange elements of $\{\omega_p\}_{p=1}^{\infty} \cup \{\lambda_q\}_{q=1}^{\infty}$ in increasing order with counting multiplicity of ω_p or λ_q and denote $\{\mu_n\}_{n=1}^{\infty}$.*

Using this notation, we state one of the main results in this paper.

Theorem 1.6 *For each $n \in \mathbb{N}$, $\lim_{\zeta \to 0} \mu_n(\zeta) = \mu_n$.*

The above theorem will be proved in Section 2 by using the mini-max principle. From now we will give more precise approximation formulae. To formulate the approximation, we prepare the following notation.

Notation 1.7 *The natural numbers $p(k)$, $q(k)$ and $n(k)$ $(k = 1, 2, \dots)$ are defined inductively by*

$$\begin{aligned} p(1) &= 1, \quad p(k+1) = \min\{p \in \mathbb{N} : \omega_p > \omega_{p(k)}\}, \\ q(1) &= 1, \quad q(k+1) = \min\{q \in \mathbb{N} : \lambda_q > \lambda_{q(k)}\}, \\ n(1) &= 1, \quad n(k+1) = \min\{n \in \mathbb{N} : \mu_n > \mu_{n(k)}\}. \end{aligned}$$

Let the natural numbers $P(k)$, $Q(k)$ and $N(k)$ $(k = 1, 2, \dots)$ imply the multiplicities of $\omega_{p(k)}$, $\lambda_{q(k)}$ and $\mu_{n(k)}$ respectively.

That is, $P(k) = p(k+1) - p(k)$, $Q(k) = q(k+1) - q(k)$ and $N(k) = n(k+1) - n(k)$. It is clear that if $\mu_{n(k)} = \omega_{p(k')} = \lambda_{q(k'')} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$ then $N(k) = P(k') + Q(k'')$. If $\mu_{n(k)} = \omega_{p(k')} \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$ then $N(k) = P(k')$ and so on. Next we introduce functions U_p on Ω_2 for some ω_p and V_q on Ω_1 for some λ_q.

Notation 1.8 *For $\omega_p \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, we define U_p by the unique solution to the equation*

$$\Delta U_p + \omega_p U_p = 0 \text{ in } \Omega_2, \quad U_p = \phi_p \text{ on } \Gamma_2.$$

For $\lambda_q \in \{\lambda_q\}_{q=1}^{\infty} \setminus \{\omega_p\}_{p=1}^{\infty}$, we define V_q by the unique solution to the equation

$$\Delta V_q + \lambda_q V_q = 0 \text{ in } \Omega_1, \quad \frac{\partial V_q}{\partial \nu_1} + \frac{\partial \psi_q}{\partial \nu_2} = 0 \text{ on } \Gamma_2, \quad V_q = 0 \text{ on } \Gamma_1$$

where ν_2 is the unit outward normal vector on $\partial\Omega_2$, that is, $\nu_2 = -\nu_1$ on Γ_2.

Using these functions ϕ_p, ψ_q, U_p and V_q, we define matrices $\boldsymbol{A}_k$, $\boldsymbol{B}_k$, $\boldsymbol{C}_k$ and $\widetilde{\boldsymbol{C}}_k$ below to state main results.

Notation 1.9 *For $\mu_{n(k)} = \omega_{p(k')} \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, the $N(k)$ square matrix $\boldsymbol{A}_k$ is defined by*

$$\boldsymbol{A}_k = \left(\int_{\Gamma_2} U_{p(k')+i-1} \frac{\partial U_{p(k')+j-1}}{\partial \nu_2} dS_x \right)_{1 \le i,j \le N(k)}.$$

Notation 1.10 *For $\mu_{n(k)} = \lambda_{q(k')} \in \{\lambda_q\}_{q=1}^{\infty} \setminus \{\omega_p\}_{p=1}^{\infty}$, the $N(k)$ square matrix $\boldsymbol{B}_k$ is defined by*

$$\boldsymbol{B}_k = \left(- \int_{\Gamma_2} V_{q(k')+i-1} \frac{\partial V_{q(k')+j-1}}{\partial \nu_1} dS_x \right)_{1 \le i,j \le N(k)}.$$

Notation 1.11 *For $\mu_{n(k)} = \omega_{p(k')} = \lambda_{q(k'')} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, the $P(k') \times Q(k'')$ matrix $\boldsymbol{C}_k$ is defined by*

$$\boldsymbol{C}_k = \left(\int_{\Gamma_2} \phi_{p(k')+i-1} \frac{\partial \psi_{q(k'')+j-1}}{\partial \nu_2} dS_x \right)_{1 \le i \le P(k'), 1 \le j \le Q(k'')}$$

and the $N(k)$ symmetric matrix $\widetilde{\boldsymbol{C}}_k$ is defined by

$$\widetilde{\boldsymbol{C}}_k = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{C}_k \\ {}^t\boldsymbol{C}_k & \boldsymbol{O} \end{pmatrix}.$$

Using the above matrices, we state the main results in this paper.

Theorem 1.12 *Assume $\mu_n \notin \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, then there exists*

$$\lim_{\zeta \to 0} \frac{\mu_n(\zeta) - \mu_n}{\zeta}.$$

This value is denoted by $\mu_n^{(1)}$ and these values are characterized as follows: (i) *if $\mu_n = \mu_{n(k)} \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, then $\mu_{n(k)}^{(1)}, \ldots, \mu_{n(k+1)-1}^{(1)}$ are the eigenvalues of the matrix $\boldsymbol{A}_k$.* (ii) *If $\mu_n = \mu_{n(k)} \in \{\lambda_q\}_{q=1}^{\infty} \setminus \{\omega_p\}_{p=1}^{\infty}$, then $\mu_{n(k)}^{(1)}, \ldots, \mu_{n(k+1)-1}^{(1)}$ are the eigenvalues of the matrix $\boldsymbol{B}_k$.*

Theorem 1.13 *Assume $\mu_n \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, then there exists*

$$\lim_{\zeta \to 0} \frac{\mu_n(\zeta) - \mu_n}{\zeta^{1/2}}.$$

This value is denoted by $\mu_n^{(1/2)}$ and if $\mu_n = \mu_{n(k)} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$ then the limits $\mu_{n(k)}^{(1/2)}, \ldots, \mu_{n(k+1)-1}^{(1/2)}$ are the eigenvalues of the matrix $\widetilde{\boldsymbol{C}}_k$.

The above theorems will be proved in Section 3. We remark here that the eigenvalues of the matrices $\boldsymbol{A}_k$, $\boldsymbol{B}_k$ and $\widetilde{\boldsymbol{C}}_k$ are well defined. It can be checked in the following simple argument. When a set $\{\hat{\phi}_p\}_{p=1}^{\infty}$ is another system of orthonormalized eigenfunctions of (1.4), let $\hat{U}_p$ be the function defined in Notation 1.8 by replacing ϕ_p with $\hat{\phi}_p$ and $\hat{\boldsymbol{A}}_k$ the symmetric matrix defined in Notation 1.9 by replacing U_p with $\hat{U}_p$. The functions $\hat{U}_p$ satisfy

$$\hat{U}_p = \sum_{i=p(k)}^{p(k+1)-1} (\hat{\phi}_p, \phi_i)_{L^2(\Omega_1)} U_i$$

for $\omega_{p(k)} = \omega_p$ and the matrix

$$\boldsymbol{P} = \Big((\hat{\phi}_{p(k)+i-1}, \phi_{p(k)+j-1})_{L^2(\Omega_1)}\Big)_{1\le i,j\le P(k)}$$

is an orthogonal matrix. Since $\boldsymbol{P}\hat{\boldsymbol{A}}_k{}^t\boldsymbol{P} = \boldsymbol{A}_k$, the eigenvalues of $\hat{\boldsymbol{A}}_k$ are equal to the eigenvalues of $\boldsymbol{A}_k$. Similarly, the eigenvalues of $\boldsymbol{B}_k$ and $\widetilde{\boldsymbol{C}}_k$ are well defined.

We also note that the eigenvalues of $\widetilde{\boldsymbol{C}}_k$ are $0, \dots, 0, \pm\kappa_1, \dots, \pm\kappa_r$ where κ_i is an eigenvalue of the matrix ${}^t\boldsymbol{C}_k\boldsymbol{C}_k$ or $\boldsymbol{C}_k{}^t\boldsymbol{C}_k$ and $r = l(k)$ is defined below. If some eigenvalues of $\widetilde{\boldsymbol{C}}_k$ are 0, we have a more precise approximation formula.

Notation 1.14 *For $\mu_{n(k)} = \omega_{p(k')} = \lambda_{q(k'')} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, let $\Lambda_i(k)$ $(i = 1, \dots, N(k))$ be the eigenvalues arranged in increasing order (counting multiplicity) of the matrix $\widetilde{\boldsymbol{C}}_k$ and $\{(\boldsymbol{u}_i, \boldsymbol{v}_i)\}_{i=1}^{N(k)}$ the set of orthonormalized eigenvectors where*

$$\boldsymbol{u}_i = (u_{i\,1}, \dots, u_{i\,P(k')}), \quad \boldsymbol{v}_i = (v_{i\,1}, \dots, v_{i\,Q(k'')}).$$

That is, the vector $(\boldsymbol{u}_i, \boldsymbol{v}_i)$ satisfies

$$(\boldsymbol{u}_i, \boldsymbol{v}_i)\begin{pmatrix} \boldsymbol{O} & \boldsymbol{C}_k \\ {}^t\boldsymbol{C}_k & \boldsymbol{O} \end{pmatrix} = \Lambda_i(k)\,(\boldsymbol{u}_i, \boldsymbol{v}_i)\,. \tag{1.9}$$

Let $l(k) = \operatorname{rank}({}^t\boldsymbol{C}_k\boldsymbol{C}_k) = \operatorname{rank}(\boldsymbol{C}_k{}^t\boldsymbol{C}_k)$ and $L(k) = N(k) - 2l(k)$. If $L(k) \ge 1$, we set for $s = 1, \dots, L(k)$

$$\phi_{k,s} = \sum_{j=1}^{P(k')} u_{i\,j}\,\phi_{p(k')+j-1}, \quad \psi_{k,s} = \sum_{j=1}^{Q(k'')} v_{i\,j}\,\psi_{q(k'')+j-1}, \quad (i = l(k) + s).$$

We define $U_{k,s}$ by a solution to

$$\Delta U_{k,s} + \mu_{n(k)} U_{k,s} = 0 \ \textit{in}\ \Omega_2, \quad U_{k,s} = \phi_{k,s}\ \textit{on}\ \Gamma_2 \tag{1.10}$$

and $V_{k,s}$ by a solution to

$$\begin{cases} \Delta V_{k,s} + \mu_{n(k)} V_{k,s} = 0 \ \textit{in}\ \Omega_1, \\ \dfrac{\partial V_{k,s}}{\partial \nu_1} + \dfrac{\partial \psi_{k,s}}{\partial \nu_2} = 0 \ \textit{on}\ \Gamma_2, \quad V_{k,s} = 0 \ \textit{on}\ \Gamma_1. \end{cases} \tag{1.11}$$

We remark that the eigenvalues $\Lambda_{l(k)+s}(k) = 0$ $(1 \leq s \leq L(k))$ and the remainders $\Lambda_i(k) \neq 0$. Thus we have

$$\int_{\Gamma_1} \phi_p \frac{\partial \psi_{k,s}}{\partial \nu_2}\, dS_x = 0 \quad (p(k') \leq p \leq p(k'+1) - 1) \tag{1.12}$$

$$\int_{\Gamma_2} \phi_{k,s} \frac{\partial \psi_q}{\partial \nu_2}\, dS_x = 0 \quad (q(k'') \leq q \leq q(k''+1) - 1) \tag{1.13}$$

by (1.9) and hence equations (1.10) and (1.11) have some solutions.

Notation 1.15 *For $\mu_{n(k)} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$ and $L(k) \geq 1$, the $L(k)$ square matrix $\boldsymbol{D}_k$ is defined by*

$$\boldsymbol{D}_k = \left(\int_{\Gamma_2} \left(\frac{\partial U_{k,i}}{\partial \nu_2} U_{k,j} - V_{k,i} \frac{\partial V_{k,j}}{\partial \nu_1} \right) dS_x \right)_{1 \leq i,j \leq L(k)}.$$

It is easy to see that the matrix $\boldsymbol{D}_k$ is well-defined since (1.12) and (1.13) hold. Using the above matrix we have the following.

Theorem 1.16 *Assume $\mu_{n(k)} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$ and $L(k) \geq 1$. Then for $n = n(k) + l(k) + j - 1, j = 1, \ldots, L(k)$ there exist*

$$\lim_{\zeta \to 0} \frac{\mu_n(\zeta) - \mu_n}{\zeta}.$$

This limit is denoted by $\mu_n^{(1)}$ and these limits $\mu_{n(k)+l(k)+j-1}^{(1)}$ $(1 \leq j \leq L(k))$ are the eigenvalues of the matrix $\boldsymbol{D}_k$.

2 APPROXIMATION OF EIGENVALUES

We give a characterization of the asymptotics of the eigenvalues (justification of Theorem 1.6). For that purpose we prepare several Lemmas. We begin with estimations of the eigenvalues from above.

Lemma 2.1 *For each $n \in \mathbb{N}$, there exist constants $M_n > 0$ and $\zeta^* > 0$ such that*

$$\mu_n(\zeta) \leq \mu_n + M_n \zeta^{1/2} \quad (0 < \zeta < \zeta^*).$$

Proof. We define certain approximate eigenfunctions $\Phi^{(0)}_{n,\zeta} \in H^1_0(\Omega)$ in the following way. Case (i). If $\mu_n = \mu_{n(k)} = \omega_{p(k')} \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, we set $p = p(k') + n - n(k)$ and

$$\Phi^{(0)}_{n,\zeta} = \begin{cases} \phi_p, & \text{in } \overline{\Omega}_1, \\ U_p, & \text{in } \Omega_2. \end{cases}$$

Case (ii). If $\mu_n = \mu_{n(k)} = \lambda_{q(k')} \in \{\lambda_q\}_{q=1}^{\infty} \setminus \{\omega_p\}_{p=1}^{\infty}$, we set $q = q(k') + n - n(k)$ and

$$\Phi^{(0)}_{n,\zeta} = \begin{cases} 0, & \text{in } \overline{\Omega}_1, \\ \zeta^{-1/2}\,\psi_q, & \text{in } \Omega_2. \end{cases}$$

Case (iii). If $\mu_n = \mu_{n(k)} = \omega_{p(k')} = \lambda_{q(k'')} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, we set $i = 1 + n - n(k)$ and

$$\Phi^{(0)}_{n,\zeta} = \begin{cases} \sum\limits_{j=1}^{P(k')} u_{i\,j}\,\phi_{p(k')+j-1}, & \text{in } \overline{\Omega}_1, \\ \zeta^{-1/2} \sum\limits_{j=1}^{Q(k'')} v_{i\,j}\,\psi_{q(k'')+j-1} + W, & \text{in } \Omega_2 \end{cases}$$

where $(\boldsymbol{u}_i, \boldsymbol{v}_i) = (u_{i\,1}, \ldots, u_{i\,P(k')}, v_{i\,1}, \ldots, v_{i\,Q(k'')})$ is the i-th eigenvector of the matrix $\tilde{\boldsymbol{C}}_k$ and W is the unique solution to the following boundary value problem

$$\Delta W = 0 \text{ in } \Omega_2, \qquad W = \sum_{j=1}^{P(k')} u_{i\,j}\,\phi_{p(k')+j-1} \text{ on } \Gamma_2.$$

A simple calculation shows with Notations 1.3 to 1.5 and 1.8

$$\int_\Omega \Phi^{(0)}_{n,\zeta}\,\Phi^{(0)}_{m,\zeta}\,a_\zeta\,dx = \delta_{n\,m} + O(\zeta^{1/2}), \tag{2.1}$$

$$\int_\Omega \nabla\Phi^{(0)}_{n,\zeta}\nabla\Phi^{(0)}_{m,\zeta}\,a_\zeta\,dx = \mu_n\,\delta_{n\,m} + O(\zeta^{1/2}). \tag{2.2}$$

Let Y_n be a subspace of $H^1_0(\Omega)$ spanned by a set $\{\Phi^{(0)}_{1,\zeta}, \ldots, \Phi^{(0)}_{n,\zeta}\}$ which becomes n-dimensional for small $\zeta > 0$ (cf. (2.1)). For any $(n-1)$-dimensional subspace $Y \subset H^1_0(\Omega)$, there exists an element $\Phi \in Y_n$ such that $\Phi \not\equiv 0$ and $\Phi \perp^\zeta Y$. The element Φ is given by $\Phi = \sum_{j=1}^{n} \alpha_j \Phi^{(0)}_{j,\zeta}$ with $\sum_{j=1}^{n} \alpha_j^2 \neq 0$. From (2.1) and (2.2), we obtain $R_\zeta(\Phi) \leq \mu_n + O(\zeta^{1/2})$. Applying the mini-max principle (1.4), we obtain the estimation Lemma 2.1.

Notation 2.2 *We set functions* $\Psi_{n,\zeta} = \zeta^{1/2}\Phi_{n,\zeta}$ *for* $n \geq 1$.

By the aid of the upper estimate of $\mu_n(\zeta)$, we study the limit of the eigenfunctions $\Phi_{n,\zeta}$ in two cases (not necessary to disjoint).

Lemma 2.3 *Suppose* $\liminf_{\zeta\to 0}\zeta^{1/2}\|\Phi_{n,\zeta}\|_{L^2(\Omega_2)} > 0$. *Let* $\{\zeta_l\}_{l=1}^\infty$ *be a positive sequence such that* $\zeta_l \to 0$ *as* $l \to \infty$. *Then there exist a subsequence* $\{\tilde{\zeta}_l\}_{l=1}^\infty \subset \{\zeta_l\}_{l=1}^\infty$, *a constant* $\tilde{\mu}_n \in \mathbb{R}$ *and* $\Psi_n \in C^0(\Omega) \cap H_0^1(\Omega)$ *such that*

$$\begin{cases} \Psi_{n,\tilde{\zeta}_l} \to \Psi_n \textit{ weak in } H_0^1(\Omega) \textit{ as } l \to \infty, \\ \Psi_{n,\tilde{\zeta}_l} \to \Psi_n \textit{ strong in } L^2(\Omega) \textit{ as } l \to \infty, \\ \mu_n(\tilde{\zeta}_l) \to \tilde{\mu}_n \textit{ as } l \to \infty, \end{cases} \tag{2.3}$$

$$\begin{cases} \Delta\Psi_n + \tilde{\mu}_n\Psi_n = 0 \textit{ in } \Omega_2, \quad \Psi_n = 0 \textit{ in } \overline{\Omega}_1, \\ \|\Psi_n\|_{L^2(\Omega_2)} > 0, \quad \tilde{\mu}_n \in \{\lambda_p\}_{p=1}^\infty. \end{cases} \tag{2.4}$$

Proof. Let $\delta = \liminf_{\zeta\to 0}\zeta\|\Phi_{n,\zeta}\|^2_{L^2(\Omega_2)}$. Without loss of generality, the sequence $\{\zeta_l\}_{l=1}^\infty$ satisfies $0 < \zeta_l \le 1$ and $\|\Psi_{n,\zeta_l}\|^2_{L^2(\Omega_1)} \ge \delta/2$ for any $l \ge 1$. Put $M_n' = \mu_n + M_n$. The functions Ψ_{n,ζ_l} and Φ_{n,ζ_l} satisfy $\|\Phi_{n,\zeta_l}\|^2_{L^2(\Omega_1)} + \|\Psi_{n,\zeta_l}\|^2_{L^2(\Omega_2)} = 1$ and $\|\nabla\Phi_{n,\zeta_l}\|^2_{L^2(\Omega_1)} + \|\nabla\Psi_{n,\zeta_l}\|^2_{L^2(\Omega_2)} = \mu_n(\zeta_l)$ for (1.5) and (1.6). Then we have

$$\begin{aligned} &\|\Psi_{n,\zeta_l}\|^2_{L^2(\Omega)} \le 1, \quad \|\nabla\Psi_{n,\zeta_l}\|^2_{L^2(\Omega)} \le M_n', \\ &\delta/2 \le \|\Psi_{n,\zeta_l}\|^2_{L^2(\Omega_2)} \le 1, \quad \|\Psi_{n,\zeta_l}\|^2_{L^2(\Omega_1)} \le \zeta_l \text{ as } l \to \infty \end{aligned}$$

and $0 \le \mu_n(\zeta_l) \le M_n'$ by Lemma 2.1. According to the Rellich theorem, there exist a subsequence $\{\tilde{\zeta}_l\}_{l=1}^\infty \subset \{\zeta_l\}_{l=1}^\infty$ and $\Psi_n \in H_0^1(\Omega)$ and $\tilde{\mu}_n \in \mathbb{R}$ such that (2.3) and $\|\Psi_n\|_{L^2(\Omega_2)} > 0$ and $\|\Psi_n\|_{L^2(\Omega_1)} = 0$. For $\varphi \in H_0^1(\Omega_2)$, we have

$$\int_{\Omega_2}\left(\nabla\Psi_{n,\tilde{\zeta}_l}\nabla\varphi - \mu_n(\tilde{\zeta}_l)\,\Psi_{n,\tilde{\zeta}_l}\,\varphi\right)dx = 0$$

and consequently

$$\int_{\Omega_2}(\nabla\Psi_n\nabla\varphi - \tilde{\mu}_n\,\Psi_n\,\varphi)\,dx = 0.$$

Thus $\Psi_n|_{\Omega_2} \in C^2(\Omega_2)$ and $\Delta\Psi_n + \tilde{\mu}_n\Psi_n = 0$ in Ω_2.

Next we want to check the continuity of Ψ_n across Γ_2. Let $\Sigma(\rho)$ be a neighborhood of Γ_2 defined by $\Sigma(\rho) = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma_2) < \rho\}$ for a small $\rho \in (0, \rho')$. Here ρ' is a constant such that $\Sigma(\rho') \Subset \Omega$ and for any $x \in \Sigma(\rho')$ there exists a unique point $x_0 \in \Gamma_2$ with $\operatorname{dist}(x, x_0) = \operatorname{dist}(x, \Gamma_2)$. Applying standard interior and boundary estimates we have $\|\Psi_{n,\tilde{\zeta}_l}\|_{L^\infty(\Omega_1\setminus\Sigma(\rho))} \to 0$ as

$l \to \infty$ and there exists a constant c_1 such that $\|\Psi_{n,\zeta_l}\|_{L^\infty(\Omega_2\setminus\Sigma(\rho))} < c_1$ for any $l \geq 1$. Let h be a function on $\Sigma(\rho)$ defined by

$$h(x) = \begin{cases} \operatorname{dist}(x, \Gamma_2) & \text{for } x \in \overline{\Omega}_1 \cap \Sigma(\rho), \\ -\operatorname{dist}(x, \Gamma_2) & \text{for } x \in \Omega_2 \cap \Sigma(\rho). \end{cases}$$

From a simple calculation, we have $\Delta(\varphi\circ h)(x) = (\varphi''\circ h)(x) + \Delta h(x)\,(\varphi'\circ h)(x)$ $(x \in \Sigma(\rho))$ where φ is a C^2 function and $\varphi \circ h$ implies a composite function $\varphi(h(x))$. Let η be a constant with $\eta > (M_n')^{1/2}$. Let α, β and ρ be constants $\alpha = \eta - \sqrt{\eta^2 - M_n'}$, $\beta = \eta + \sqrt{\eta^2 - M_n'}$ and $\rho = (\beta-\alpha)^{-1}\log\{(\beta+1)/(\alpha+1)\}$ respectively. We take η such that $0 < \rho < \rho'$ and $2\eta > \sup\{|\Delta h(x)| : x \in \Sigma(\rho')\}$. Let $\theta_{1,\zeta}$ and $\theta_{2,\zeta}$ be continuous functions on the interval $[-\rho, \rho]$ defined by

$$\theta_{1,\zeta}(t) = \frac{\beta e^{\alpha t} - \alpha e^{\beta t}}{\beta e^{\alpha\rho} - \alpha e^{\beta\rho}} \quad (-\rho \leq t \leq \rho),$$

$$\theta_{2,\zeta}(t) = \begin{cases} \dfrac{(1+\alpha)e^{\beta\rho}}{\beta - \alpha}\left((1+\beta\zeta)e^{\alpha t} - (1+\alpha\zeta)e^{\beta t}\right) & (-\rho \leq t < 0), \\ \dfrac{\zeta(1+\alpha)e^{\beta\rho}}{\beta - \alpha}\left((1+\beta)e^{\alpha t} - (1+\alpha)e^{\beta t}\right) & (0 \leq t \leq \rho). \end{cases}$$

We set

$$\theta_\zeta(t) = b_1(\zeta)\,\theta_{1,\zeta}(t) + b_2(\zeta)\,\theta_{2,\zeta}(t) \quad (-\rho \leq t \leq \rho)$$

where $b_1(\zeta) = \zeta + \|\Psi_{n,\zeta}\|_{L^\infty(\Omega_1\setminus\Sigma(\rho))}$ and $b_2(\zeta) = 1 + \|\Psi_{n,\zeta}\|_{L^\infty(\Omega_2\setminus\Sigma(\rho))}$. Direct calculation gives for $\zeta < (2M_n')^{-1}$

$$\begin{cases} \theta_\zeta \in C([-\rho,\rho]), \quad \theta_\zeta(0) = (b_1(\zeta) + \zeta\, b_2(\zeta)\, e^{2\eta\rho})\,(1+\alpha)\, e^{-\alpha\rho}, \\ \theta_\zeta(-\rho) \geq b_2(\zeta), \quad \theta_\zeta(t) > 0 \quad (-\rho \leq t \leq \rho), \quad \theta_\zeta(\rho) = b_1(\zeta), \\ \theta_\zeta'(t) \leq -1/2 + b_1(\zeta)\, M_n' \quad (-\rho < t < 0), \quad \theta_\zeta'(t) \leq 0 \quad (0 < t < \rho), \\ \lim\limits_{t\uparrow 0} \zeta\theta_\zeta'(t) - \lim\limits_{t\downarrow 0} \theta_\zeta'(t) = 0, \quad \theta_\zeta''(t) - 2\eta\,\theta_\zeta'(t) + M_n'\,\theta_\zeta(t) = 0 \quad (t \neq 0). \end{cases}$$

We set

$$\Theta_\zeta(x) = \theta_\zeta(h(x)) \quad \text{for } x \in \Sigma(\rho).$$

Then $(\Psi_{n,\zeta} - \Theta_\zeta)/\Theta_\zeta \in H^1(\Sigma(\rho)) \cap C^2(\Sigma(\rho)\setminus\Gamma_2) \cap C(\overline{\Sigma(\rho)})$ and

$$\sup_{x\in\partial\Sigma(\rho)} \frac{\Psi_{n,\zeta}(x) - \Theta_\zeta(x)}{\Theta_\zeta(x)} \leq 0$$

and for $\varphi \in H^1(\Sigma(\rho))$

$$\nabla\left(\frac{\Psi_{n,\zeta} - \Theta_\zeta}{\Theta_\zeta}\right)\nabla\varphi - \frac{2\nabla\Theta_\zeta}{\Theta_\zeta}\nabla\left(\frac{\Psi_{n,\zeta} - \Theta_\zeta}{\Theta_\zeta}\right)\varphi$$

$$= \nabla\Psi_{n,\zeta}\nabla\left(\frac{\varphi}{\Theta_\zeta}\right) - \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\nabla\Theta_\zeta\nabla\varphi - \nabla\left(\frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\right)\nabla\Theta_\zeta\,\varphi.$$

On the other hand

$$\int_{\Sigma(\rho)\cap\Omega_1} \nabla\left(\frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\right)\nabla\Theta_\zeta\,\varphi\,dx$$
$$= \int_{\Gamma_2} \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\frac{\partial\Theta_\zeta}{\partial\nu_1}\varphi\,dS_x - \int_{\Sigma(\rho)\cap\Omega_1} \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}(\Delta\Theta_\zeta\,\varphi + \nabla\Theta_\zeta\nabla\varphi)\,dx$$

and hence

$$\int_{\Sigma(\rho)\cap\Omega_1} \nabla\left(\frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\right)\nabla\Theta_\zeta\,\varphi + \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\nabla\Theta_\zeta\nabla\varphi\,dx$$
$$= -\int_{\Gamma_2} \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\lim_{t\downarrow 0}\theta'_\zeta(t)\,\varphi\,dS_x$$
$$- \int_{\Sigma(\rho)\cap\Omega_1} \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\left((2\eta + \Delta h)\,(\theta'_\zeta\circ h)(x) - M'_n\Theta_\zeta\right)\varphi\,dx.$$

Similarly

$$\int_{\Sigma(\rho)\cap\Omega_2} \nabla\left(\frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\right)\nabla\Theta_\zeta\,\varphi + \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\nabla\Theta_\zeta\nabla\varphi\,dx$$
$$= \int_{\Gamma_2} \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\lim_{t\uparrow 0}\theta'_\zeta(t)\,\varphi\,dS_x$$
$$- \int_{\Sigma(\rho)\cap\Omega_2} \frac{\Psi_{n,\zeta}}{(\Theta_\zeta)^2}\left((2\eta + \Delta h)\,(\theta'_\zeta\circ h)(x) - M'_n\Theta_\zeta\right)\varphi\,dx.$$

Consequently for $\varphi \in C_0^1(\Sigma(\rho))$ with $\varphi \geq 0$

$$\int_{\Sigma(\rho)} \Bigg(\nabla\left(\frac{\Psi_{n,\zeta} - \Theta_\zeta}{\Theta_\zeta}\right)\nabla\varphi - \frac{2\nabla\Theta_\zeta}{\Theta_\zeta}\nabla\left(\frac{\Psi_{n,\zeta} - \Theta_\zeta}{\Theta_\zeta}\right)\varphi$$
$$- \frac{(2\eta + \Delta h)\,(\theta'_\zeta\circ h) + (\mu_n(\zeta) - M_n)\,\Theta_\zeta}{\Theta_\zeta}\left(\frac{\Psi_{n,\zeta} - \Theta_\zeta}{\Theta_\zeta}\right)\varphi\Bigg)\,a_\zeta\,dx$$
$$= \int_{\Sigma(\rho)} \frac{(2\eta + \Delta h)\,(\theta'_\zeta\circ h) + (\mu_n(\zeta) - M_n)\,\Theta_\zeta}{\Theta_\zeta}\varphi\,a_\zeta\,dx \leq 0.$$

According to Theorem 8.1 in [6] we have $\Psi_{n,\zeta}(x) \leq \Theta_\zeta(x)$ for $x \in \Sigma(\rho)$. Similarly $-\Theta_\zeta(x) \leq \Psi_{n,\zeta}(x)$ for $x \in \Sigma(\rho)$. Therefore we obtain

$$|\Psi_{n,\zeta}(x)| \leq \begin{cases} b_1(\zeta) & x \in \Omega_1 \setminus \Sigma(\rho), \\ \Theta_\zeta(x) & x \in \Sigma(\rho), \\ b_2(\zeta) & x \in \Omega_2 \setminus \Sigma(\rho) \end{cases}$$

and hence $\Psi_n \in C^0(\Omega)$ and $\Psi_n(x) = 0$ $(x \in \overline{\Omega_1})$.

Lemma 2.4 *Suppose that* $\liminf_{\zeta\to 0}\|\Phi_{n,\zeta}\|_{L^2(\Omega_1)} > 0$. *Let* $\{\zeta_l\}_{l=1}^{\infty}$ *be a positive sequence such that* $\zeta_l \to 0$ *as* $l\to\infty$. *Then there exist a subsequence* $\{\tilde{\zeta}_l\}_{l=1}^{\infty} \subset \{\zeta_l\}_{l=1}^{\infty}$, *a constant* $\tilde{\mu}_n \in \mathbb{R}$ *and* $\Phi_n \in H_0^1(\Omega_1\cup\partial\Omega_2)$ *such that*

$$\begin{cases} \Phi_{n,\tilde{\zeta}_l}|_{\Omega_1} \to \Phi_n \text{ weak in } H_0^1(\Omega_1\cup\Gamma_2) \text{ as } l\to\infty, \\ \Phi_{n,\tilde{\zeta}_l}|_{\Omega_1} \to \Phi_n \text{ strong in } L^2(\Omega_1) \text{ as } l\to\infty, \\ \mu_m(\tilde{\zeta}_l) \to \tilde{\mu}_n \text{ as } l\to\infty, \end{cases} \tag{2.5}$$

$$\begin{cases} \Delta\Phi_n + \tilde{\mu}_n\Phi_n = 0 \text{ in } \Omega_1, \quad \partial\Phi_n/\partial\nu_1 = 0 \text{ on } \Gamma_2, \\ \Phi_n = 0 \text{ on } \partial\Omega, \quad \|\Phi_n\|_{L^2(\Omega_1)} > 0, \quad \tilde{\mu}_n \in \{\omega_p\}_{p=1}^{\infty}, \end{cases} \tag{2.6}$$

where $H_0^1(\Omega_1\cup\Gamma_2)$ *denotes the closure of* $C_0^\infty(\Omega_1\cup\Gamma_2)$ *in* $H^1(\Omega_1)$.

Proof. Let $\delta = \liminf\limits_{\zeta\to 0}\|\Phi_{m,\zeta}\|_{L^2(\Omega_1)}$ and a positive sequence $\{\zeta_l\}_{l=0}^{\infty}$ satisfy that $\zeta_l \le 1$ and $\|\Phi_{n,\zeta_l}\|_{L^2(\Omega_1)} \ge \delta/2$ for any $l\ge 1$. From (1.5) and Lemma 2.1, we have

$$\|\nabla\Phi_{n,\zeta_l}\|_{L^2(\Omega_1)}^2 \le M_n', \quad 0 \le \mu_n(\zeta_l) \le M_n'.$$

There exist a subsequence $\{\tilde{\zeta}_l\}_{l=1}^{\infty} \subset \{\zeta_l\}_{l=1}^{\infty}$, $\tilde{\mu}_n \in \mathbb{R}$ and $\Phi_n \in H^1(\Omega_1)$ such that (2.5) and $\|\Phi_n\|_{L^2(\Omega_1)} > 0$ by the Rellich theorem. From (1.5) and (1.6), we have $\zeta\|\Phi_{m,\zeta}\|_{L^2(\Omega_2)}^2 \le 1$ and $\zeta\|\nabla\Phi_{m,\zeta}\|_{L^2(\Omega_2)}^2 \le M_n'$ and hence for $\zeta = \tilde{\zeta}_l$ and $\varphi \in H_0^1(\Omega)$

$$\begin{aligned} &\left|\int_{\Omega_1} \nabla\Phi_{n,\zeta}\nabla\varphi - \mu_n(\zeta)\,\Phi_{n,\zeta}\,\varphi\,dx\right| \\ &= \left|\zeta\int_{\Omega_2} \nabla\Phi_{n,\zeta}\nabla\varphi - \mu_n(\zeta)\,\Phi_{n,\zeta}\,\varphi\,dx\right| \\ &\le \zeta\left\{\|\nabla\Phi_{n,\zeta}\|_{L^2(\Omega_2)}\|\nabla\varphi\|_{L^2(\Omega_2)} + \mu_n(\zeta)\|\Phi_{n,\zeta}\|_{L^2(\Omega_2)}\|\varphi\|_{L^2(\Omega_2)}\right\} \\ &\le \zeta^{1/2}\{(M_n')^{1/2} + M_n'\}\,\|\varphi\|_{H^1(\Omega_2)}. \end{aligned}$$

Let $l\to\infty$, we have

$$\int_{\Omega_1} \nabla\Phi_m\nabla\varphi - \tilde{\mu}_m\,\Phi_m\,\varphi\,dx = 0 \quad \text{for } \varphi \in H_0^1(\Omega).$$

This implies (2.6) and we complete the proof of Lemma 2.4.

Note that either of the condition of Lemma 2.3 or Lemma 2.4 holds for any $n\in\mathbb{N}$.

Lemma 2.5 *Let $\{\zeta_l\}_{l=1}^{\infty}$ be a positive sequence such that $\zeta_l \to 0$ as $l \to \infty$. Then there exist a subsequence $\{\tilde{\zeta}_l\}_{l=1}^{\infty} \subset \{\zeta_l\}_{l=1}^{\infty}$ and functions $\Phi_n \in L^2(\Omega_1)$, $\Psi_n \in L^2(\Omega_2)$ and constants $\tilde{\mu}_n$ $(n = 1, 2, \dots)$ such that*

$$\begin{cases} \Psi_{n,\tilde{\zeta}_l}|_{\Omega_2} \to \Psi_n \text{ in } L^2(\Omega_2), & \Phi_{n,\tilde{\zeta}_l}|_{\Omega_1} \to \Phi_n \text{ in } L^2(\Omega_1), \\ \mu_n(\tilde{\zeta}_l) \to \tilde{\mu}_n \text{ as } l \to \infty, & \tilde{\mu}_n \in \{\omega_p\}_{p=1}^{\infty} \cup \{\lambda_q\}_{q=1}^{\infty}, \end{cases} \tag{2.7}$$

$$\int_{\Omega_1} \Phi_n \Phi_m \, dx + \int_{\Omega_2} \Psi_n \Psi_m \, dx = \delta_{nm} \quad \text{for } n, m \geq 1. \tag{2.8}$$

Moreover, if $\tilde{\mu}_n \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, the limit Φ_n is an eigenfunction of (1.7) *and $\Psi_n \equiv 0$ in Ω_2 and if $\tilde{\mu}_n \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, the limit Ψ_n is an eigenfunction of* (1.8) *and $\Phi_n \equiv 0$ in Ω_1.*

Proof. The functions $\Phi_{n,\zeta}$ and $\Psi_{n,\zeta}$ satisfy

$$\begin{aligned} &\|\Phi_{n,\zeta}\|_{L^2(\Omega_1)} \leq 1, \quad \|\Psi_{n,\zeta}\|_{L^2(\Omega_2)} \leq 1, \\ &\|\nabla\Phi_{n,\zeta}\|_{L^2(\Omega_1)} \leq M_n', \quad \|\nabla\Psi_{n,\zeta}\|_{L^2(\Omega_2)} \leq M_n', \\ &\int_{\Omega_1} \Phi_{n,\zeta} \Phi_{m,\zeta} \, dx + \int_{\Omega_2} \Psi_{n,\zeta} \Psi_{m,\zeta} \, dx = \delta_{nm}. \end{aligned} \tag{2.9}$$

From $\|\Phi_{1,\zeta}\|_{L^2(\Omega_1)} \geq 0$, the lower limit of $\|\Phi_{1,\zeta}\|_{L^2(\Omega_1)}$ as $\zeta \to 0$ is 0 or $\delta > 0$. If $\liminf_{\zeta\to 0} \|\Phi_{1,\zeta}\|_{L^2(\Omega_1)} = 0$, by taking a subsequence, we have $\lim_{l\to\infty} \|\Psi_{1,\zeta_l}\|_{L^2(\Omega_2)} = 1$. Hence we have $\tilde{\mu}_1 \in \{\lambda_q\}_{q=1}^{\infty}$ from Lemma 2.3. If $\liminf_{\zeta\to 0} \|\Phi_{1,\zeta}\|_{L^2(\Omega_1)} = \delta$, by taking a subsequence, we have $\tilde{\mu}_1 \in \{\omega_p\}_{p=1}^{\infty}$ by Lemma 2.4. From the Rellich theorem and the above arguments, there exists a subsequence $\{\zeta(1,l)\}_{l=1}^{\infty} \subset \{\zeta_l\}_{l=1}^{\infty}$ such that

$$\begin{cases} \Psi_{1,\zeta(1,l)}|_{\Omega_2} \to \Psi_1 \text{ in } L^2(\Omega_2) \text{ as } l \to \infty, \\ \Phi_{1,\zeta(1,l)}|_{\Omega_1} \to \Phi_1 \text{ in } L^2(\Omega_1) \text{ as } l \to \infty, \\ \mu_1(\zeta(1,l)) \to \tilde{\mu}_1 \text{ as } l \to \infty, \quad \tilde{\mu}_1 \in \{\omega_p\}_{p=1}^{\infty} \cup \{\lambda_q\}_{q=1}^{\infty}. \end{cases}$$

Inductively, by taking a subsequence $\{\zeta(n,l)\}_{l=1}^{\infty} \subset \{\zeta(n-1,l)\}_{l=1}^{\infty}$ for $n \geq 2$, we have also

$$\begin{cases} \Psi_{n,\zeta(n,l)}|_{\Omega_2} \to \Psi_n \text{ in } L^2(\Omega_2) \text{ as } l \to \infty, \\ \Phi_{n,\zeta(n,l)}|_{\Omega_1} \to \Phi_n \text{ in } L^2(\Omega_1) \text{ as } l \to \infty, \\ \mu_n(\zeta(n,l)) \to \tilde{\mu}_n \text{ as } l \to \infty, \quad \tilde{\mu}_n \in \{\omega_p\}_{p=1}^{\infty} \cup \{\lambda_q\}_{q=1}^{\infty} \end{cases}$$

and (2.8) for $1 \leq m \leq n$ by (2.9). We apply the diagonal argument to this situation. Namely, by setting $\tilde{\zeta}_l = \zeta(l,l)$ $(l = 1, 2, \dots)$, we obtain (2.7) and (2.8) for any $n \in \mathbb{N}$.

It is obvious that $\Psi_n \equiv 0$ in Ω_2 for $\tilde{\mu}_n \notin \{\lambda_q\}_{q=1}^\infty$ and that $\Phi_n \equiv 0$ in Ω_1 for $\tilde{\mu}_n \notin \{\omega_k\}_{k=1}^\infty$. Thus we complete the proof of Lemma 2.5.

Proof of Theorem 1.6. Using the mini-max principle and Lemma 2.5, we have

$$\tilde{\mu}_n \le \tilde{\mu}_{n+1}, \quad \tilde{\mu}_n \le \mu_n, \quad \tilde{\mu}_n \in \{\mu_n\}_{n=1}^\infty = \{\omega_p\}_{p=1}^\infty \cup \{\lambda_q\}_{q=1}^\infty.$$

Clearly $\tilde{\mu}_1 = \mu_1$. We assume that $\tilde{\mu}_1 = \mu_1, \dots, \tilde{\mu}_n = \mu_n$. Then we have

$$\mu_n \le \tilde{\mu}_{n+1} \le \mu_{n+1}.$$

If $\mu_n = \mu_{n+1}$, we have $\tilde{\mu}_{n+1} = \mu_{n+1}$ immediately. If $\mu_n < \mu_{n+1}$, we have $\tilde{\mu}_{n+1} = \mu_n$ or $\tilde{\mu}_{n+1} = \mu_{n+1}$. Let k be the number with $\mu_{n(k)} = \mu_n$. Then $n(k+1) = n+1$. If $\mu_n = \tilde{\mu}_{n+1} \in \{\omega_p\}_{p=1}^\infty \setminus \{\lambda_q\}_{q=1}^\infty$, we have

$$\int_{\Omega_1} \Phi_i \Phi_j \, dx = \delta_{ij} \quad \text{for } n(k) \le i, j \le n(k+1)$$

by Lemma 2.5. This means that the dimension of the eigenspace of $\mu_{n(k)}$ is greater than or equal to $n(k+1) - n(k) + 1$. This is contrary to that the multiplicity of $\mu_{n(k)}$ is $N(k) = n(k+1) - n(k)$. Similarly, if $\mu_n = \tilde{\mu}_{n+1} \in \{\lambda_q\}_{q=1}^\infty \setminus \{\omega_p\}_{p=1}^\infty$, we have a contradiction to the multiplicity of $\mu_{n(k)}$. If $\mu_n = \tilde{\mu}_{n+1} \in \{\lambda_q\}_{q=1}^\infty \cap \{\omega_p\}_{p=1}^\infty$, we have

$$\int_{\Omega_1} \Phi_i \Phi_j \, dx + \int_{\Omega_2} \Psi_i \Psi_j \, dx = \delta_{ij} \quad \text{for } n(k) \le i, j \le n(k+1)$$

by Lemma 2.5. Let k' and k'' be the numbers with $\omega_{p(k')} = \lambda_{q(k'')} = \mu_{n(k)} = \mu_n$ and we set

$$\boldsymbol{a}_i = (a_{i\,1}, \dots, a_{i\,P(k')}), \quad a_{i\,j} = \int_{\Omega_1} \Phi_i \, \phi_{p(k')+j-1} \, dx, \quad j = 1, \dots, P(k'),$$

$$\boldsymbol{b}_i = (b_{i\,1}, \dots, b_{i\,Q(k'')}), \quad b_{i\,j} = \int_{\Omega_2} \Psi_i \, \psi_{q(k'')+j-1} \, dx, \quad j = 1, \dots, Q(k'').$$

We have

$$(\boldsymbol{a}_i, \boldsymbol{b}_i) \cdot (\boldsymbol{a}_j, \boldsymbol{b}_j) = \delta_{ij} \quad \text{for } n(k) \le i, j \le n(k+1).$$

This implies that $\{(\boldsymbol{a}_i, \boldsymbol{b}_i)\}_{i=n(k)}^{n(k+1)}$ is an orthonormal basis of $\mathbb{R}^{N(k)}$. This contradicts that the dimension of $\mathbb{R}^{N(k)}$ is $N(k)$. Thus $\tilde{\mu}_{n+1} \ne \mu_n$ and hence $\tilde{\mu}_{n+1} = \mu_{n+1}$. Since the set $\{\mu_n\}_{n=1}^\infty$ is independent of a choice of sequences $\{\zeta_l\}_{l=1}^\infty$, we obtain Theorem 1.6.

3 A PRECISE APPROXIMATION OF EIGENVALUES

We derive a precise characterization of asymptotics of the eigenvalues $\mu_n(\zeta)$ for ζ. For that purpose, we will construct accurate approximate eigenfunctions. For simple notation, we set

$$(\phi,\psi)_1 = \int_{\Omega_1} \phi\,\psi\,dx \quad \text{and} \quad (\phi,\psi)_2 = \int_{\Omega_2} \phi\,\psi\,dx.$$

Proof of Theorem 1.12. First we deal with the case (i). For $\omega_p \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, we define $U_p^{(1)}$ by the unique solution to the boundary value problem

$$\Delta U_p^{(1)} = 0 \text{ in } \Omega_1, \quad \frac{\partial U_p^{(1)}}{\partial \nu_1} + \frac{\partial U_p}{\partial \nu_2} = 0 \text{ on } \Gamma_2, \quad U_p^{(1)} = 0 \text{ on } \Gamma_1,$$

and $U_p^{(2)}$ is defined by the unique solution to

$$\Delta U_p^{(2)} = 0 \text{ in } \Omega_2, \quad U_p^{(2)} = U_p^{(1)} \text{ on } \Gamma_2.$$

We set

$$\phi_{p,\zeta} = \begin{cases} \phi_p + \zeta U_p^{(1)} & \text{in } \overline{\Omega}_1, \\ U_p + \zeta U_p^{(2)} & \text{in } \Omega_2. \end{cases}$$

Using matching conditions on Γ_2 for $\phi_p, U_p, U_p^{(1)}$ and $U_p^{(2)}$, we have $\phi_{p,\zeta} \in H_0^1(\Omega)$. If $\mu_n = \omega_p \in \{\omega_p\}_{p=1}^{\infty} \setminus \{\lambda_q\}_{q=1}^{\infty}$, we substitute $\phi_{p,\zeta}$ for φ in (1.5) and we have

$$\begin{aligned} &\frac{\mu_n(\zeta) - \mu_n}{\zeta} \left((\Phi_{n,\zeta}, \phi_p)_1 + \zeta (\Phi_{n,\zeta}, U_p)_2 \right) \\ &= -\mu_n(\zeta)(\Phi_{n,\zeta}, U_p^{(1)})_1 + \zeta \left((\nabla\Phi_{n,\zeta}, \nabla U_p^{(2)})_2 - \mu_n(\zeta)(\Phi_{n,\zeta}, U_p^{(2)})_2 \right). \quad (3.1) \end{aligned}$$

From Theorem 1.6 and Lemma 2.5, the limit Φ_n is expressed by

$$\Phi_n = \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1\, \phi_p \quad \text{in } \Omega_1$$

for k' with $\mu_n = \omega_{p(k')}$. Hence

$$\frac{\mu_n(\zeta) - \mu_n}{\zeta} \left((\Phi_{n,\zeta}, \Phi_n)_1 + \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1\, \zeta\, (\Phi_{n,\zeta}, U_p)_2 \right)$$

$$= -\mu_n(\zeta) \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1 \, (\Phi_{n,\zeta}, U_p^{(1)})_1$$

$$+ \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1 \, \zeta \left((\nabla\Phi_{n,\zeta}, \nabla U_p^{(2)})_2 - \mu_n(\zeta)(\Phi_{n,\zeta}, U_p^{(2)})_2 \right).$$

On the other hand, the eigenfunction $\Phi_{n,\zeta}$ satisfies

$$\zeta(\Phi_{n,\zeta}, U_p)_2 = O(\zeta^{1/2}),$$
$$\zeta \left((\nabla\Phi_{n,\zeta}, \nabla U_p^{(2)})_2 - \mu_n(\zeta)(\Phi_{n,\zeta}, U_p^{(2)})_2 \right) = O(\zeta^{1/2}).$$

Put $\zeta = \widetilde{\zeta}_l$ and take $l \to \infty$, we get

$$\lim_{l\to\infty} \frac{\mu_n(\zeta) - \mu_n}{\zeta} = -\mu_n \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1 \, (\Phi_n, U_p^{(1)})_1.$$

We denote this value by $\mu_n^{(1)}$. It will be proved that $\mu_n^{(1)}$ dose not depend on the choice of the original sequence $\{\zeta_l\}_{l=1}^{\infty}$ (well-defined). We consider the limit of $l \to \infty$ for $\zeta = \widetilde{\zeta}_l$ in (3.1) and we have

$$\begin{aligned} \mu_n^{(1)}(\Phi_n, \phi_p)_1 &= -\mu_n(\Phi_n, U_p^{(1)})_1 = (\Delta\Phi_n, U_p^{(1)})_1 \\ &= \int_{\Gamma_2} \Phi_n \frac{\partial U_p}{\partial \nu_2} \, dS_x \\ &= \sum_{i=1}^{P(k')} (\Phi_n, \phi_{p(k')+i-1})_1 \int_{\Gamma_2} U_{p(k')+i-1} \frac{\partial U_p}{\partial \nu_2} \, dS_x \end{aligned}$$

for $n = n(k), \ldots, n(k+1)-1$ and $p = p(k'), \ldots, p(k'+1)-1$. Hence

$$\begin{pmatrix} \mu_{n(k)}^{(1)} & & O \\ & \ddots & \\ O & & \mu_{n(k+1)-1}^{(1)} \end{pmatrix} \boldsymbol{P} = \boldsymbol{P}\boldsymbol{A}_k,$$

for an orthogonal matrix $\boldsymbol{P} = \left((\Phi_{n(k)+i-1}, \phi_{p(k')+j-1})_1 \right)_{1\le i,j\le N(k)}$. Thus we have obtained that $\mu_n^{(1)}$ $(n(k) \le n \le n(k+1)-1)$ are eigenvalues of the matrix $\boldsymbol{A}_k$.

Next we deal with the case (ii). For $\lambda_q \in \{\lambda_q\}_{q=1}^{\infty} \setminus \{\omega_p\}_{p=1}^{\infty}$, we define $V_q^{(2)}$ by the unique solution to the boundary value problem

$$\Delta V_q^{(1)} = 0 \text{ in } \Omega_2, \quad V_q^{(1)} = V_q \text{ on } \Gamma_2.$$

$$\psi_{q,\zeta} = \begin{cases} \zeta V_q & \text{in } \overline{\Omega}_1, \\ \psi_q + \zeta V_q^{(1)} & \text{in } \Omega_2. \end{cases}$$

Using matching conditions on Γ_2 for ψ_q, V_q and $V_q^{(1)}$, we have $\psi_{q,\zeta} \in H_0^1(\Omega)$. If $\mu_n = \lambda_q \in \{\lambda_q\}_{q=1}^{\infty} \setminus \{\omega_p\}_{p=1}^{\infty}$, we substitute $\psi_{q,\zeta}$ for φ in (1.5) and we have

$$\begin{aligned} &\frac{\mu_n(\zeta) - \mu_n}{\zeta} \left((\Psi_{n,\zeta}, V_q)_1 + (\Psi_{n,\zeta}, \psi_q)_2 \right) \\ &= (\nabla \Psi_{n,\zeta}, \nabla V_q^{(1)})_2 - \mu_n(\zeta)\, (\Psi_{n,\zeta}, V_q^{(1)})_2. \end{aligned} \tag{3.2}$$

From Theorem 1.6 and Lemma 2.5, the limit Ψ_n is expressed in Ω_2 by

$$\Psi_n = \sum_{q=q(k')}^{q(k'+1)-1} (\Psi_n, \psi_q)_2\, \psi_q \quad \text{in } \Omega_2$$

for k' with $\mu_n = \lambda_{q(k')}$. Hence

$$\begin{aligned} &\frac{\mu_n(\zeta) - \mu_n}{\zeta} \left(\sum_{q=q(k')}^{q(k'+1)-1} (\Psi_n, \psi_q)_2\, (\Psi_{n,\zeta}, V_q)_1 + (\Psi_{n,\zeta}, \Psi_n)_2 \right) \\ &= \sum_{q=q(k')}^{q(k'+1)-1} (\Psi_n, \psi_q)_2 \left((\nabla \Psi_{n,\zeta}, \nabla V_q^{(1)})_2 - \mu_n(\zeta)(\Psi_{n,\zeta}, V_q^{(1)})_2 \right). \end{aligned}$$

We see that the eigenfunction $\Psi_{n,\zeta}$ satisfies $(\Psi_{n,\zeta}, V_q)_1 = O(\zeta^{1/2})$ as $\zeta \to 0$. Put $\zeta = \widetilde{\zeta}_l$ and take $l \to \infty$, we get

$$\lim_{l \to \infty} \frac{\mu_n(\zeta) - \mu_n}{\zeta} = \sum_{q=q(k')}^{q(k'+1)-1} (\Psi_n, \psi_q)_2 \left((\nabla \Psi_n, \nabla V_q^{(1)})_2 - \mu_n\, (\Psi_n, V_q^{(1)})_2 \right).$$

We denote this value by $\mu_n^{(1)}$. It will be proved below that $\mu_n^{(1)}$ are well-defined. We consider $l \to \infty$ for $\zeta = \widetilde{\zeta}_l$ in (3.2) and we have

$$\begin{aligned} \mu_n^{(1)}\, (\Psi_n, \psi_q)_2 &= (\nabla \Psi_n, \nabla V_q^{(1)})_2 - \mu_n\, (\Psi_n, V_q^{(1)})_2 \\ &= \int_{\Gamma_2} \frac{\partial \Psi_n}{\partial \nu_2} V_q\, dS_x \\ &= - \sum_{j=q(k')}^{q(k'+1)-1} (\Psi_n, \psi_j)_2 \int_{\Gamma_2} V_q \frac{\partial V_j}{\partial \nu_1}\, dS_x \end{aligned}$$

for $n = n(k), \ldots, n(k+1) - 1$ and $q = q(k'), \ldots, q(k'+1) - 1$. Hence

$$\boldsymbol{P}\begin{pmatrix} \mu^{(1)}_{n(k)} & & \boldsymbol{O} \\ & \ddots & \\ \boldsymbol{O} & & \mu^{(1)}_{n(k+1)-1} \end{pmatrix} = \boldsymbol{B}_k \boldsymbol{P}$$

for an orthogonal matrix $\boldsymbol{P} = \left((\psi_{q(k')+i-1}, \Psi_{n(k)+j-1})_2 \right)_{1 \le i,j \le N(k)}$. This means that $\mu^{(1)}_n$ $(n(k) \le n \le n(k+1) - 1)$ are the eigenvalues of the matrix $\boldsymbol{B}_k$.

Since the eigenvalues of these matrices are independent of a choice of sequences $\{\zeta_l\}_{l=1}^{\infty}$, we obtain Theorem 1.12.

Proof of Theorem 1.13. The same manner to that of the construction of approximate eigenfunctions used in the above proofs can not be applied to the case where $\mu_n \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, since U_p and V_q can not be defined in Notation 1.8 if $\omega_p = \lambda_q$.

For $\omega_p \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, we define U_p by the unique solution to the boundary value problem

$$\Delta U_p = 0 \text{ in } \Omega_2, \quad U_p = \phi_p \text{ on } \Gamma_2.$$

We set

$$\varphi_p = \begin{cases} \phi_p & \text{in } \overline{\Omega}_1, \\ U_p & \text{in } \Omega_2. \end{cases}$$

If $\mu_n = \omega_p \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, we substitute φ_p for φ in (1.5) and we have

$$\frac{\mu_n(\zeta) - \mu_n}{\zeta^{1/2}} (\Phi_{n,\zeta}, \phi_p)_1 = (\nabla \Psi_{n,\zeta}, \nabla U_p)_2 - \mu_n(\zeta)\, (\Psi_{n,\zeta}, U_p)_2. \tag{3.3}$$

For $\lambda_p \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$ we define V_q by the unique solution to the boundary value problem

$$\Delta V_q = 0 \text{ in } \Omega_1, \quad \frac{\partial V_q}{\partial \nu_1} + \frac{\partial \psi_q}{\partial \nu_2} = 0 \text{ on } \Gamma_2, \quad V_q = 0 \text{ on } \Gamma_1$$

and $V^{(1)}_q$ is defined by the unique solution to the boundary value problem

$$\Delta V^{(1)}_q = 0 \text{ in } \Omega_2, \quad V^{(1)}_q = V_q \text{ in } \Gamma_2.$$

We set

$$W_{q,\zeta} = \begin{cases} \zeta V_q & \text{in } \overline{\Omega}_1, \\ \psi_q + \zeta V^{(1)}_q & \text{on } \Omega_2. \end{cases}$$

If $\mu_n = \lambda_q \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, we substitute $W_{q,\zeta}$ for φ in (1.5) and we have

$$\begin{aligned}
&\frac{\mu_n(\zeta) - \mu_n}{\zeta^{1/2}} (\Psi_{n,\zeta}, \psi_q)_2 \\
&= -\mu_n(\zeta)\,(\Phi_{n,\zeta}, V_q)_1 + \zeta^{1/2}\left((\nabla\Psi_{n,\zeta}, \nabla V_q^{(1)})_2 - \mu_n(\zeta)\,(\Psi_{n,\zeta}, V_q^{(1)})_2\right). \quad (3.4)
\end{aligned}$$

If $\mu_n = \omega_{p(k')} = \lambda_{q(k'')} \in \{\omega_p\}_{p=1}^{\infty} \cap \{\lambda_q\}_{q=1}^{\infty}$, the limits Φ_n and Ψ_n are expressed by

$$\Phi_n = \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1\, \phi_p \ \text{ in } \Omega_1, \qquad \Psi_n = \sum_{q=q(k'')}^{q(k''+1)-1} (\Psi_n, \psi_q)_2\, \psi_q \ \text{ in } \Omega_2$$

by Theorem 1.6 and Lemma 2.5. We have

$$\begin{aligned}
&\frac{\mu_n(\zeta) - \mu_n}{\zeta^{1/2}} \left((\Phi_{n,\zeta}, \Phi_n)_1 + (\Psi_{n,\zeta}, \Phi_n)_2\right) \\
&= \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1 \left((\nabla\Psi_{n,\zeta}, \nabla U_p)_2 - \mu_n(\zeta)\,(\Psi_{n,\zeta}, U_p)_2\right) \\
&\quad - \mu_n(\zeta) \sum_{q=q(k'')}^{q(k''+1)-1} (\Psi_n, \psi_q)_2\, (\Phi_{n,\zeta}, V_q)_1 \\
&\quad + \zeta^{1/2} \sum_{q=q(k'')}^{q(k''+1)-1} (\Psi_n, \psi_q)_2 \left((\nabla\Psi_{n,\zeta}, \nabla V_q^{(1)})_2 - \mu_n(\zeta)\,(\Psi_{n,\zeta}, V_q^{(1)})_2\right).
\end{aligned}$$

Put $\zeta = \tilde{\zeta}_l$ and take $l \to \infty$, we get

$$\begin{aligned}
\lim_{l\to\infty} \frac{\mu_n(\zeta) - \mu_n}{\zeta^{1/2}} &= \sum_{p=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_p)_1 \left((\nabla\Psi_n, \nabla U_p)_2 - \mu_n\,(\Psi_n, U_p)_2\right) \\
&\quad - \mu_n \sum_{q=q(k'')}^{q(k''+1)-1} (\Psi_n, \psi_q)_2\, (\Phi_n, V_q)_1
\end{aligned}$$

for n with $\mu_n = \mu_{n(k)}$. We denote this value by $\mu_n^{(1/2)}$. It will be proved below that $\mu_n^{(1/2)}$ are well-defined. We consider $l \to \infty$ for $\zeta = \tilde{\zeta}_l$ in (3.3) and we have

$$\begin{aligned}
\mu_n^{(1/2)}(\Phi_n, \phi_p)_1 &= (\nabla\Psi_n, \nabla U_p)_2 - \mu_n\,(\Psi_n, U_p)_2 \\
&= \int_{\Gamma_2} \frac{\partial \Psi_n}{\partial \nu_2}\, \phi_p\, dS_x
\end{aligned}$$

$$= \sum_{j=q(k'')}^{q(k''+1)-1} (\Psi_n, \psi_j)_2 \int_{\Gamma_2} \phi_p \frac{\partial \psi_j}{\partial \nu_2} dS_x$$

for $n = n(k), \ldots, n(k+1) - 1$ and $p = p(k'), \ldots, p(k'+1) - 1$. Similarly, we consider $l \to \infty$ for $\zeta = \zeta_l$ in (3.4) and we have

$$\begin{aligned}
\mu_n^{(1/2)}(\Psi_n, \psi_q)_2 &= -\mu_n (\Phi_n, V_q)_1 \\
&= \int_{\Gamma_2} \Phi_n \frac{\partial \psi_q}{\partial \nu_2} dS_x \\
&= \sum_{i=p(k')}^{p(k'+1)-1} (\Phi_n, \phi_i)_1 \int_{\Gamma_2} \phi_i \frac{\partial \psi_q}{\partial \nu_2} dS_x
\end{aligned}$$

for $n = n(k), \ldots, n(k+1) - 1$ and $q = q(k''), \ldots, q(k''+1) - 1$. Thus

$$\begin{pmatrix} \mu_{n(k)}^{(1/2)} & & \boldsymbol{O} \\ & \ddots & \\ \boldsymbol{O} & & \mu_{n(k+1)-1}^{(1/2)} \end{pmatrix} \boldsymbol{P} = \boldsymbol{P} \widetilde{\boldsymbol{C}}_k$$

for an orthogonal matrix $\boldsymbol{P}$ given by

$$\begin{aligned}
\boldsymbol{P} &= (\boldsymbol{P}_1 \ \boldsymbol{P}_2), \\
\boldsymbol{P}_1 &= \left((\Phi_{n(k)+i-1}, \phi_{p(k')+j-1})_1\right)_{1 \le i \le N(k),\, 1 \le j \le P(k')}, \\
\boldsymbol{P}_2 &= \left((\Psi_{n(k)+i-1}, \psi_{q(k'')+j-1})_2\right)_{1 \le i \le N(k),\, 1 \le j \le Q(k'')}.
\end{aligned}$$

This means that $\mu_n^{(1/2)}$ $(n(k) \le n \le n(k+1) - 1)$ are the eigenvalues of the matrix $\widetilde{\boldsymbol{C}}_k$. Since these eigenvalues are independent of a choice of sequences $\{\zeta_l\}_{l=1}^{\infty}$, we obtain Theorem 1.13.

Proof of Theorem 1.16. For $s = 1, \ldots, L(k)$, we define $V_{k,s}^{(1)}$ by the unique solution to

$$\Delta V_{k,s}^{(1)} = 0 \text{ in } \Omega_2, \quad V_{k,s}^{(1)} = V_{k,s} \text{ on } \Gamma_2,$$

and $U_{k,s}^{(1)}$ by

$$\Delta U_{k,s}^{(1)} = 0 \text{ in } \Omega_1, \quad \frac{\partial U_{k,s}^{(1)}}{\partial \nu_1} + \frac{\partial U_{k,s}}{\partial \nu_2} = 0 \text{ on } \Gamma_2, \quad U_{k,s}^{(1)} = 0 \text{ on } \Gamma_1,$$

and $U_{k,s}^{(2)}$ by

$$\Delta U_{k,s}^{(2)} = 0 \text{ in } \Omega_2, \quad U_{k,s}^{(2)} = U_{k,s}^{(1)} \text{ on } \Gamma_2.$$

We set an approximate eigenfunction

$$\varphi_{k,s}=\begin{cases}\phi_{k,s}+\zeta^{1/2}V_{k,s}+\zeta U_{k,s}^{(1)} & \text{in } \overline{\Omega}_1,\\ \zeta^{-1/2}\psi_{k,s}+U_{k,s}+\zeta^{1/2}V_{k,s}^{(1)}+\zeta U_{k,s}^{(2)} & \text{in } \Omega_2.\end{cases}$$

Using matching conditions on Γ_2, we have $\varphi_{k,s}\in H_0^1(\Omega)$. We substitute $\varphi_{k,s}$ for φ in (1.5) and we get

$$\begin{aligned}&\frac{\mu_n(\zeta)-\mu_n}{\zeta}\left((\Phi_{n,\zeta},\phi_{k,s})_1+(\Psi_{n,\zeta},\psi_{k,s})_2\right)\\ &=\frac{\mu_n(\zeta)-\mu_n}{\zeta^{1/2}}\left((\Phi_{n,\zeta},V_{k,s})_1+(\Psi_{n,\zeta},U_{k,s})_2\right)\\ &\quad-\mu_n(\zeta)(\Phi_{n,\zeta},U_{k,s}^{(1)})_1+(\nabla\Psi_{n,\zeta},\nabla V_{k,s}^{(1)})_2-\mu_n(\zeta)(\Psi_{n,\zeta},V_{k,s}^{(1)})_2\\ &\quad+\zeta^{1/2}\left((\nabla\Psi_{n,\zeta},\nabla U_{k,s}^{(2)})_2-\mu_n(\zeta)(\Psi_{n,\zeta},U_{k,s}^{(2)})_2\right).\end{aligned}\tag{3.5}$$

On the other hand, we set

$$\begin{aligned}\tilde{\boldsymbol{u}}_n&=\left((\Phi_n,\phi_{p(k')})_1,\dots,(\Phi_n,\phi_{p(k'+1)-1})_1\right),\\ \tilde{\boldsymbol{v}}_n&=\left((\Psi_n,\psi_{q(k'')})_2,\dots,(\Psi_n,\psi_{p(k''+1)-1})_2\right).\end{aligned}$$

Then for $n=n(k)+l(k)+j-1$, $1\le j\le L(k)$, the vectors $(\tilde{\boldsymbol{u}}_n,\tilde{\boldsymbol{v}}_n)$ are eigenvectors of the eigenvalue 0 of $\widetilde{C}_k$ and we have

$$(\tilde{\boldsymbol{u}}_n,\tilde{\boldsymbol{v}}_n)=\sum_{s=1}^{L(k)}b_{n\,s}(\boldsymbol{u}_{l(k)+s},\boldsymbol{v}_{l(k)+s})$$

where $b_{n\,s}=\tilde{\boldsymbol{u}}_n{}^t\boldsymbol{u}_{l(k)+s}+\tilde{\boldsymbol{v}}_n{}^t\boldsymbol{v}_{l(k)+s}$. Hence

$$\Phi_n=\sum_{s=1}^{L(k)}b_{n\,s}\phi_{k,s}\ \text{ in } \Omega_1,\quad \Psi_n=\sum_{s=1}^{L(k)}b_{n\,s}\psi_{k,s}\ \text{ in } \Omega_2$$

for $n=n(k)+l(k)+j-1$, $1\le j\le L(k)$ and we have

$$\begin{aligned}&\frac{\mu_n(\zeta)-\mu_n}{\zeta}\left((\Phi_{n,\zeta},\Phi_n)_1+(\Psi_{n,\zeta},\Psi_n)_2\right)\\ &=\sum_{s=1}^{L(k)}b_{n\,s}\left(-\mu_n(\zeta)\,(\Phi_{n,\zeta},U_{k,s}^{(1)})_1+(\nabla\Psi_{n,\zeta},\nabla V_{k,s}^{(1)})_2-\mu_n(\zeta)\,(\Psi_{n,\zeta},V_{k,s}^{(1)})_2\right)\\ &\quad+\frac{\mu_n(\zeta)-\mu_n}{\zeta^{1/2}}\sum_{s=1}^{L(k)}b_{n\,s}\left((\Phi_{n,\zeta},V_{k,s})_1+(\Psi_{n,\zeta},U_{k,s})_2\right)+O(\zeta^{1/2}).\end{aligned}$$

Put $\zeta = \tilde{\zeta}_l$ and take $l \to \infty$, then for $n = n(k) + l(k) + j - 1, 1 \le j \le L(k)$ there exist

$$\lim_{l\to\infty} \frac{\mu_n(\zeta) - \mu_n}{\zeta}$$
$$= \sum_{s=1}^{L(k)} b_{ns} \left(-\mu_n \, (\Phi_n, U_{k,s}^{(1)})_1 + (\nabla\Psi_n, \nabla V_{k,s}^{(1)})_2 - \mu_n \, (\Psi_n, V_{k,s}^{(1)})_2 \right).$$

We denote this value by $\mu_n^{(1)}$. It will be proved below that $\mu_n^{(1)}$ are well-defined. We consider $l \to \infty$ for $\zeta = \tilde{\zeta}_l$ in (3.5) and we get

$$\begin{aligned} \mu_n^{(1)} b_{ns} &= -\mu_n(\Phi_n, U_{k,s}^{(1)})_1 + (\nabla\Psi_n, \nabla V_{k,s}^{(1)})_2 - \mu_n(\Psi_n, V_{k,s}^{(1)})_2 \\ &= \int_{\Gamma_2} \left(\Phi_n \frac{\partial U_{k,s}}{\partial \nu_2} + \frac{\partial \Psi_n}{\partial \nu_2} V_{k,s} \right) dS_x \\ &= \sum_{j=1}^{L(k)} b_{nj} \int_{\Gamma_2} \left(U_{k,j} \frac{\partial U_{k,s}}{\partial \nu_2} - \frac{\partial V_{k,j}}{\partial \nu_1} V_{k,s} \right) dS_x \end{aligned}$$

for $n = n(k) + l(k) + j - 1$, $1 \le j, s \le L(k)$. Hence

$$\boldsymbol{P} \begin{pmatrix} \mu_{n(k)+l(k)}^{(1)} & & \boldsymbol{O} \\ & \ddots & \\ \boldsymbol{O} & & \mu_{n(k+1)-l(k)-1}^{(1)} \end{pmatrix} = \boldsymbol{D}_k \boldsymbol{P}$$

for an orthogonal matrix $\boldsymbol{P}$ given by

$$\boldsymbol{P} = (b_{ni})_{1\le i,j \le L(k)} \quad (n = n(k) + l(k) + j - 1).$$

This means that the limits $\mu_{n(k)+l(k)}^{(1)}, \cdots, \mu_{n(k+1)-l(k)-1}^{(1)}$ are the eigenvalues of the matrix $\boldsymbol{D}_k$. Since these eigenvalues are independent of a choice of sequences $\{\zeta_l\}_{l=1}^{\infty}$, we obtain Theorem 1.16.

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